ON THE DETERMINATION OF THE ELASTIC POTENTIAL FROM EXPERIMENTS

(OB OPREDELENII UPRUGIKH POTENTSIALOV IZ OPYTA)

PMM Vol.25, No.4, 1961, pp. 750-751

M. E. EGLIT
(Moscow)

(Received April 23, 1961)

In simple tension Hooke's law may be written in the following form:

$$\sigma = E\varepsilon \tag{1}$$

Here σ is the tensile stress, ϵ is the relative elongation, and E is a constant – Young's modulus.

It has been shown by Ivlev [1] that in the case of small deformations the validity of the law (1) in simple tension is insufficient for a generalization of a linear Hooke's law to arbitrary small deformations. He has constructed examples of a physically nonlinear elastic body which in simple tension obeys the same law (1).

It is known [2] that within the framework of the theory of finite elastic deformations it is, in general, impossible to construct an isotropic elastic medium in which the components of the stress tensor depend linearly on the components of the finite strain tensor in a space of initial states. Moreover, this statement is valid not only for arbitrary deformations but even for particular classes of deformation such as plane strain and plane stress. In spite of this, it is possible to construct examples of nonlinear elastic bodies for which Equation (1) is valid for finite deformations under simple tension.

Within the framework of small deformation theory (geometrically linear theory of elasticity) we shall show that the fact that Hooke's law is satisfied for arbitrary plane strain or plane stress of the material likewise does not allow one to conclude that the material satisfies Hooke's law in an arbitrary spatial state of deformation.

An elastic medium may be specified by the assignment of the free energy $F'(\epsilon_{ij}, T)$ or the thermodynamic potential $\psi'(\sigma^{ij}/\rho, T)$, where ρ is the density and T is the temperature of the medium. The connection between the stresses and the strains then has the form [3]

$$\sigma^{ij} = \rho \frac{\partial F'}{\partial \varepsilon_{ij}} \quad \text{or} \quad \varepsilon_{ij} = \frac{\partial \Psi'}{\partial (\sigma^{ij}/\rho)} \tag{2}$$

Formulas (2) are valid for arbitrary processes in the general case of finite deformations of elastic bodies.

If the strains are small, then for physically nonlinear bodies the density in Formulas (2) may be taken as constant and

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

In this case Formulas (2) may be represented in the form

$$\sigma_{ij} = \frac{\partial F}{\partial \varepsilon_{ij}}, \qquad \varepsilon_{ij} = \frac{\partial \Psi}{\partial \sigma_{ij}} \tag{3}$$

Clearly, Formulas (3) must be viewed as physical approximations. This circumstance must be kept in mind in investigations in which small additions to the stresses of the order of density changes are being taken into account.

For an isotropic body the first of Formulas (3) can be written in the form

$$\sigma_{ij} = \left(\frac{\partial F}{\partial I_1} + I_1 \frac{\partial F}{\partial I_2} + I_2 \frac{\partial F}{\partial I_3}\right)\delta_{ij} - \left(\frac{\partial F}{\partial I_2} + I_1 \frac{\partial F}{\partial I_3}\right)\varepsilon_{ij} + \frac{\partial F}{\partial I_3}\varepsilon_{ik}\varepsilon_{kj} \tag{4}$$

Here I_1, I_2, I_3 are the system of invariants of the strain tensor. In Cartesian coordinates we have

$$I_1 = \varepsilon_{ii}, \qquad I_2 = \frac{1}{2}(\varepsilon_{ii}\varepsilon_{jj} - \varepsilon_{ij}\varepsilon_{ji}), \qquad I_3 = |\varepsilon_{ij}| = \begin{vmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{vmatrix} \tag{5}$$

Let the material satisfy the relationship

$$\sigma_{ij} = \lambda_0 I_1 \delta_{ij} + 2\mu_0 \varepsilon_{ij} \tag{6}$$

under an isothermal process of plane strain ($T = T_0$, $I_3 = 0$).

It is obvious that an arbitrary function F, having the form

$$F = \frac{\lambda(I_1, I_2, I_3, T) + 2\mu(I_1, I_2, I_3, T)}{2} I_1^2 - 2\mu(I_1, I_2, I_3, T) I_2 + f(I_1, I_2, I_3, T) \tag{7}$$

under the conditions

$$\lambda\,(I_1, I_2, 0, T_0) = \lambda_0, \qquad \mu\,(I_1, I_2, 0, T_0) = \mu_0$$

$$\frac{\partial\lambda}{\partial I_1} = \frac{\partial\lambda}{\partial I_2} = \frac{\partial\lambda}{\partial I_3} = \frac{\partial\mu}{\partial I_1} = \frac{\partial\mu}{\partial I_2} = \frac{\partial\mu}{\partial I_3} = \frac{\partial f}{\partial I_1} = \frac{\partial f}{\partial I_2} = \frac{\partial f}{\partial I_3} = 0 \quad \text{for } I_3 = 0$$

can serve as the potential function (free energy) of such a material. In this material Hooke's law is satisfied for arbitrary plane strain, but is not satisfied in the general case.

In an analogous way, by using the second of the relations (3) and by choosing a function $\psi(\sigma_{ij}, T)$, it is easy to construct an elastic medium which for isothermal processes in plane stress satisfies Hooke's law but which does not satisfy Hooke's law in general.

BIBLIOGRAPHY

1. Ivlev, D.D., K postroeniiu teorii uprugosti (On the construction of the theory of elasticity). *Dokl. Akad. Nauk SSSR* Vol. 138, No. 6, 1961.

2. Sedov, L.I., *Osnovy nelineinoi mekhaniki sploshnoi sredy (Foundations of the Nonlinear Mechanics of Continuous Media)*. Izd. VTs AN SSSR, 1960.

3. Novozhilov, V.V., *Teoriia uprugosti (Theory of Elasticity)*. Sudpromgiz, 1958.

Translated by E.E.Z.